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Existence and unicity of solutions for a non-local relaxation equation

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Abstract

We study the following one-dimensional evolution equation:

$$\frac{\partial u}{\partial t}(x,t) = \int_{A^+u(x,t)} \lambda_1(\xi,t)\,(u(\xi,t)-u(x,t))\,\mathrm{d}\xi - \int_{A^-u(x,t)} \lambda_2(\xi,t)\,(u(x,t)-u(\xi,t))\,\mathrm{d}\xi,$$

where $A^+u(x,t) = \{\xi \in [0,1] \mid u(\xi,t) > u(x,t)\}$, $A^-u(x,t) = [0,1] \setminus A^+u(x,t)$, and λ_1, λ_2 are non-negative functions.

We prove the existence of solutions for a particular class of initial data $u(x,0)$. We also prove that the solutions are unique. Finally, under additional constraints on the initial data, we give an explicit expression for the solution.

MSC: 35K65; 47G20; 45K05

Keywords: Relaxation equation; Integro-differential equation; Existence and uniqueness

1. Introduction

Equations of the form

$$\frac{\partial u}{\partial t} = \alpha(c-u), \tag{1.1}$$

where α, c are constants or given functions, and u is an unknown function of space and time, are ubiquitous as models of relaxation phenomena. The oldest instance of such a model is Newton's law of radiative cooling, where u is to be interpreted as the temperature of an hot, optically thin body, c as the equilibrium temperature, and α as the inverse of the relaxation characteristic time. Operators having the form $\alpha(c-u)$ arise often in fluid dynamics, typically as Rayleigh friction terms (for an overview of applications see, e.g. [1]). Here we study a non-local generalization of Eq. (1.1). We shall assume $u = u(x,t)$ with $x \in [0,1]$ and $t \geq 0$. For each $x \in [0,1]$ we define the two sets

$$A^+u(x,t) = \{\xi \in [0,1] \mid u(\xi,t) > u(x,t)\}, \tag{1.2}$$

$$A^-u(x,t) = [0,1] \setminus A^+u(x,t). \tag{1.3}$$

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Next we choose two weight functions $\lambda_1, \lambda_2 : [0, 1] \times [0, +\infty) \to [0, \infty)$. We consider the following equation:

$$\frac{\partial u}{\partial t}(x,t) = \int_{A^+u(x,t)} \lambda_1(\xi,t)\,(u(\xi,t) - u(x,t))\,\mathrm{d}\xi - \int_{A^-u(x,t)} \lambda_2(\xi,t)\,(u(x,t) - u(\xi,t))\,\mathrm{d}\xi. \tag{1.4}$$

The Eq. (1.4) is justified observing that

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{b-a}\int_a^b u(\xi,t)\mathrm{d}\xi - u(x,t) \tag{1.5}$$

is of type (1.1) and remarking that

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$$\frac{1}{b-a}\int_a^b u(\xi,t)\mathrm{d}\xi - u(x,t) = \int_{A^+u(x,t)} (u(\xi,t) - u(x,t))\,\mathrm{d}\xi - \int_{A^-u(x,t)} (u(x,t) - u(\xi,t))\,\mathrm{d}\xi. \tag{1.6}$$

Then, for $b - a = 1$ Eq. (1.5) assumes the form (1.4) with $\lambda_1 = \lambda_2$. Eq. (1.5), in turn, is reminiscent of the Milne problem in radiative transfer [2]. The case $b - a \neq 1$ suggests a further generalization, as stated in problem C of Section 5.

In this paper we present theorems granting uniqueness and existence of solutions for this non-local relaxation equation, under suitable conditions on the choice of the initial data $u(x, 0) = u_0(x)$ and on λ_1 and λ_2. The main difficulty in dealing with Eq. (1.4) is controlling the time evolution of the sets $A^+u(x,t)$ and $A^-u(x,t)$, which we achieve by putting some restrictions on the admissible initial data u_0.

In the concluding remarks we will mention some open problems which we believe are interesting, although difficult, even just for the one-dimensional case treated in this note.

2. A first result

In this section we prove an existence theorem for Eq. (1.1) which requires very strong conditions on the initial data $u_0 = u(x, 0)$. This will give an example, in a simple case, of the Picard iteration technique used for the proof of the more general theorems of Section 3.

Theorem 2.1. *Let λ_1, λ_2 be two real integrable functions defined on $[0, 1] \times \mathbb{R}^+$ such that*

$$\exists M > 0 : 0 \le \lambda_i(x,t) \le M \quad \forall (x,t) \in [0,1] \times \mathbb{R}^+, i = 1, 2.$$

Let u_0 be a $C^1([0, 1])$ function for which $u_0'(x) > 0$, $\forall x \in [0, 1]$. Then we have that $\exists T_0 > 0$, $\exists u : [0, 1] \times [0, T_0] \longrightarrow \mathbb{R}$ such that $\forall (x,t) \in [0,1] \times [0, T_0]$

$$u(x,t) = u_0(x) + \int_0^t \Bigg[\int_{A^+u(x,\tau)} \lambda_1(\xi,\tau)\,(u(\xi,\tau) - u(x,\tau))\,\mathrm{d}\xi - \int_{A^-u(x,\tau)} \lambda_2(\xi,\tau)\,(u(x,\tau) - u(\xi,\tau))\,\mathrm{d}\xi\Bigg]\mathrm{d}\tau. \tag{2.1}$$

Furthermore, this solution also holds for any arbitrary time $t > T_0$.

Proof. Let us consider a sequence of functions starting from u_0, and defined by the following recurrence relation

$$u_{n+1}(x,t) = u_0(x) + \int_0^t \Bigg[\int_{A^+u_n(x,\tau)} \lambda_1(\xi,\tau)\,(u_n(\xi,\tau) - u_n(x,\tau))\,\mathrm{d}\xi - \int_{A^-u_n(x,\tau)} \lambda_2(\xi,\tau)\,(u_n(x,\tau) - u_n(\xi,\tau))\,\mathrm{d}\xi\Bigg]\mathrm{d}\tau. \tag{2.2}$$

From the monotonicity of u_0 it follows that: $\xi \in A^+u_0(x) \Leftrightarrow u_0(\xi) > u_0(x) \Leftrightarrow \xi > x$. Hence for the function u_1 we can write

$$u_1(x,t) = u_0(x) + \int_0^t \Bigg[\int_x^1 \lambda_1(\xi,\tau)\,(u_0(\xi) - u_0(x))\,\mathrm{d}\xi - \int_0^x \lambda_2(\xi,\tau)\,(u_0(x) - u_0(\xi))\,\mathrm{d}\xi\Bigg]\mathrm{d}\tau.$$

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Deriving partially with respect to x, we have the following equality

$$\frac{\partial u_1}{\partial x}(x,t) = u_0'(x)\left[1 - \int_0^t \left(\int_x^1 \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_0^x \lambda_2(\xi,\tau)\mathrm{d}\xi\right)\mathrm{d}\tau\right].$$

Because of the boundedness and non-negativity of the weight functions λ_1, λ_2, we have

$$u_0'(x)(1 - Mt) \le \frac{\partial u_1}{\partial x}(x,t) \le u_0'(x) \quad \forall (x,t) \in [0,1] \times R^+.$$

If we choose T_0 such that $0 < T_0 < \frac{1}{M}$, we have: $\frac{\partial u_1}{\partial s}(x,t) > 0$.

Then $u_1(\cdot,t)$ is a strictly growing function in $[0,1]$ for all $t \in [0,T_0]$; it follows that $A^+ u_1(x,t) = A^+ u_0(x)$. Deriving the second element of recursion (2.2), we have

$$\frac{\partial u_2}{\partial x}(x,t) = u_0'(x) - \int_0^t \frac{\partial u_1}{\partial x}(x,\tau)\left(\int_x^1 \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_0^x \lambda_2(\xi,\tau)\mathrm{d}\xi\right)\mathrm{d}\tau.$$

and, arguing in the same way

$$0 < u_0'(x)(1 - Mt) \le \frac{\partial u_2}{\partial x}(x,t) \le u_0'(x) \quad \forall (x,t) \in [0,1] \times [0,T_0],$$

and, hence: $\partial_x u_2(x,t) > 0, \forall x \in [0,1], t \in [0,T_0]$. By induction, this generalizes to

$$0 < u_0'(x)(1 - Mt) \le \frac{\partial u_n}{\partial x}(x,t) \le u_0'(x) \quad \forall (x,t) \in [0,1] \times [0,T_0],\ \forall n \in N.$$

We observe that all functions u_n are continuous in $[0,1] \times [0,T_0]$. Then we remark:

$$|u_{n+1}(x,t) - u_n(x,t)| \le \int_0^t \left[\int_x^1 2\,\|u_n - u_{n-1}\|_\infty \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_0^x 2\,\|u_n - u_{n-1}\|_\infty \lambda_2(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau.$$

Hence for every $(x,t) \in [0,1] \times [0,T_0]$:

$$|u_{n+1}(x,t) - u_n(x,t)| \le 2MT_0\,\|u_n - u_{n-1}\|_\infty.$$

So, for $0 < T_0 < \frac{1}{2M}$, the sequence $\{u_n\}_{n=0}^\infty$ is a Cauchy sequence in $[0,1] \times [0,T_0]$ with respect the Lagrangian norm and then there exists a function $u : [0,1] \times [0,T_0] \longrightarrow R$ such that $\{u_n\}_{n=0}^\infty$ is uniformly convergent to u. The function u verifies Eq. (2.1).

Finally, we notice that $u(x,T_0)$ is $C^1([0,1])$ with respect to x and that $\frac{\partial}{\partial x}u(x,t) > 0$. Furthermore, we have found that T_0 depends only on M (which, in turn, is determined by λ_i), and not on u_0. Hence the proof given so far may be iterated an arbitrary number n of times, using $u(x, nT_0)$ as the new initial data. We conclude that the solution (2.1) holds at any positive time. □

Remark 2.2. We have that:

$$\frac{\partial u_{n+1}}{\partial x}(x,t) = u_0'(x) - \int_0^t \frac{\partial u_n}{\partial x}(x,\tau)\left[\int_x^1 \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_0^x \lambda_2(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau.$$

Hence for every $n \in N$, we have:

$$\left|\frac{\partial u_{n+1}}{\partial x}(x,t) - \frac{\partial u_n}{\partial x}(x,t)\right| \le \int_0^t \left|\frac{\partial u_n}{\partial x}(x,\tau) - \frac{\partial u_{n-1}}{\partial x}(x,\tau)\right|\left[\int_x^1 \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_0^x \lambda_2(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau.$$

Then, for every $n \in N$, we have:

$$\left\|\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right\|_\infty \le MT_0 \left\|\frac{\partial u_n}{\partial x} - \frac{\partial u_{n-1}}{\partial x}\right\|_\infty$$

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so $\{\partial_x u_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and, for all $(x, t) \in [0, 1] \times [0, T_0]$:

$$\frac{\partial u}{\partial x}(x, t) = u_0'(x) - \int_0^t \frac{\partial u}{\partial x}(x, \tau) \left[\int_x^1 \lambda_1(\xi, \tau) \mathrm{d}\xi + \int_0^x \lambda_2(\xi, \tau) \mathrm{d}\xi \right] \mathrm{d}\tau. \tag{2.3}$$

Remark 2.3. For the sequence $\{u_n\}_{n=0}^{\infty}$ we have

$$|u_n(x, t)| \le \|u_0\|_\infty \sum_{i=0}^{n} \frac{(4Mt)^i}{i!}$$

and

$$0 < u_0'(x)(1 - Mt) \le \frac{\partial u_n}{\partial x}(x, t) \le u_0'(x)$$

for all $n \in \mathbb{N}$, $x \in [0, 1]$, $t \in [0, T_0]$. Hence:

$$|u_n(x, t)| \le \|u_0\|_\infty \sum_{i=0}^{n} \frac{(4Mt)^i}{i!} = \|u_0\|_\infty \, \mathrm{e}^{4Mt}$$

and

$$0 < u_0'(x)(1 - Mt) \le \frac{\partial u}{\partial x}(x, t) \le u_0'(x)$$

for $x \in [0, 1]$, $t \in [0, T_0]$.

3. The main result

In this section we consider a more general class of initial data than in Section 2. Let u_0 be a real continuous function, defined on $[0, 1]$, such that $\exists m \in N$, $\mathrm{card}\{u_0^{-1}(u_0(x))\} \le m$, $\forall x \in [0, 1]$. Define, now, $\alpha_0(x) = 0$; $\alpha_1(x) = \min\left\{\{u_0^{-1}(u_0(x))\} - \{\alpha_0(x)\}\right\}$; $\ldots$; $\alpha_j(x) = \min\left\{\{u_0^{-1}(u_0(x))\} - \{\alpha_0(x), \ldots, \alpha_{j-1}(x)\}\right\}$; $\ldots$; and $\alpha_{j(x)+1}(x) = \cdots = \alpha_{j(x)+s}(x) = 1$ if $\left\{\{u_0^{-1}(u_0(x))\} - \{\alpha_0(x), \ldots, \alpha_{j-1}(x), \ldots, \alpha_{j(x)}(x)\}\right\} = \emptyset$. We assume that u_0 and every α_j have derivative, except at a finite number of points, where they have finite left and right derivative. We also assume that u_0 has no points with zero derivative (left and right). In this situation we say that: $u_0 \in CF([0, 1])$.

Examples of such type of functions are piecewise linear functions or piecewise regular functions with a finite number of maximal and minimal points, all of which are cusps, and with no points of zero-derivative. Now we prove an existence theorem, valid for a finite interval of time, for CF initial data.

Theorem 3.1. *Assume $u_0 \in CF([0, 1])$ and let λ_1, λ_2 be two real integrable functions defined on $[0, 1] \times \mathbb{R}^+$ such that $\exists M > 0 : 0 \le \lambda_i(x, t) \le M \forall (x, t) \in [0, 1] \times \mathbb{R}^+$, $i = 1, 2$. Then $\exists T_0 > 0$, $\exists u : [0, 1] \times [0, T_0] \longrightarrow \mathbb{R}$, continuous in x and t, such that $\forall (x, t) \in [0, 1] \times [0, T_0]$*

$$u(x, t) = u_0(x, t) \int_0^t \left[\int_{A^+ u(x,\tau)} \lambda_1(\xi, \tau) \left(u(\xi, \tau) - u(x, \tau)\right) \mathrm{d}\xi \right.$$
$$\left. - \int_{A^- u(x,\tau)} \lambda_2(\xi, \tau) \left(u(x, \tau) - u(\xi, \tau)\right) \mathrm{d}\xi \right] \mathrm{d}\tau.$$

Furthermore the solution also holds for any arbitrary time $t > T_0$.

Proof. As in Theorem 2.1, let us define a sequence of functions $\{u_n\}_{n=0}^{\infty}$ by using the recurrence expression (2.2). Here the only difference is that u_0 obeys the less stringent requirement of belonging to $CF([0, 1]; \mathbb{R})$.

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For simplicity, let us consider the case $u_0'(0) > 0$ and $u_0'(1) < 0$. Then we may write the function u_1 as:

$$u_1(x,t) = u_0(x) + \int_0^t \Bigg[-\int_0^{\alpha_1(x)} \lambda_2(\xi,\tau)(u_0(x) - u_0(\xi))\mathrm{d}\xi$$
$$+ \int_{\alpha_1(x)}^{\alpha_2(x)} \lambda_1(\xi,\tau)(u_0(\xi) - u_0(x))\mathrm{d}\xi + \cdots + \int_{\alpha_{p(x)}(x)}^{1} \lambda_2(\xi,\tau)(u_0(x) - u_0(\xi))\mathrm{d}\xi \Bigg] \mathrm{d}\tau, \tag{3.1}$$

where we have expressed the domains of integration explicitly by using the cut functions $\{\alpha_j\}$ for $j = 1, \dots, p(x)$. The other cases lead to analogous expressions of what follows, and to the same final results. Deriving partially by x, we have:

$$\frac{\partial u_1}{\partial x}(x,t) = u_0'(x) - \int_0^t \left[\int_{A^+u_0(x)} \lambda_1(\xi,\tau) u_0'(x)\mathrm{d}\xi + \int_{A^-u_0(x)} \lambda_2(\xi,\tau) u_0'(x)\mathrm{d}\xi \right] \mathrm{d}\tau;$$

and, multiplying by u_0', we have:

$$u_0'(x)\frac{\partial u_1}{\partial x}(x,t) = {u_0'}^2(x)\left[1 - \int_0^t \left(\int_{A^+u_0(x)} \lambda_1(\xi,\tau)\mathrm{d}\xi + \int_{A^-u_0(x)} \lambda_2(\xi,\tau)\mathrm{d}\xi \right) \mathrm{d}\tau \right].$$

From $0 \le \lambda_i(x,t) \le M, \forall (x,t), i = 1, 2$ we obtain:

$$0 < {u_0'}^2(x)(1 - Mt) \le u_0'(x)\frac{\partial u_1}{\partial x}(x,t) \le {u_0'}^2(x) \quad \forall x, \forall t \in \left[0, \frac{1}{M}\right[. \tag{3.2}$$

We may now state the following facts:

(A) if $u_0(x_1) = u_0(x_2)$ then $u_1(x_1,t) = u_1(x_2,t), \forall t \in \left[0, \frac{1}{M}\right]$;

(B) $\forall x \in [0,1], \forall t \in \left[0, \frac{1}{M}\right[$, $u_0'(x)$ and $\partial_x u_1(x,t)$ have the same sign;

(C) if $u_0(x_1) < u_0(x_2)$ then $u_1(x_2,t) - u_1(x_1,t) \le u_0(x_2) - u_0(x_1), \forall t \in \left[0, \frac{1}{M}\right[$;

(D) $\forall x_1, x_2 \in [0,1], \forall t \in \left[0, \frac{1}{M}\right[$, $u_0(x_1) < u_0(x_2) \iff u_1(x_1,t) < u_1(x_2,t)$.

It is immediate to see that (A) follows from (3.1), and that (B) follows from (3.2). The last two statements are not immediately obvious. To prove (C) let us simplify the notation by defining

$$I_n^+(x,t) = \int_{A^+u_n(x,\tau)} \lambda_1(\xi,\tau)\,(u_n(\xi,\tau) - u_n(x,\tau))\,\mathrm{d}\xi \tag{3.3}$$

and

$$I_n^-(x,t) = \int_{A^-u_n(x,\tau)} \lambda_2(\xi,\tau)\,(u_n(x,\tau) - u_n(\xi,\tau))\,\mathrm{d}\xi. \tag{3.4}$$

From the inclusion $A^+u_0(x_2) \subset A^+u_0(x_1)$, the definition (1.3), and the fact that the integrands in (3.3) and (3.4) are non-negative, we deduce that $I_0^+(x_2,t) < I_0^+(x_1,t)$ and $I_0^-(x_2,t) > I_0^-(x_1,t)$. Then we have:

$$u_1(x_2,t) = u_0(x_2) + \int_0^t \left(I_0^+(x_2,\tau) - I_0^-(x_2,\tau) \right) \mathrm{d}\tau$$
$$\le u_0(x_2) + \int_0^t \left(I_0^+(x_1,\tau) - I_0^-(x_1,\tau) \right) \mathrm{d}\tau = u_0(x_2) - u_0(x_1) + u_1(x_1,t).$$

In order to prove (D) let us first assume $u_1(x_1,t) < u_1(x_2,t)$. This inequality may be written as

$$u_0(x_1) + \int_0^t \left(I_0^+(x_1,\tau) - I_0^-(x_1,\tau) \right) \mathrm{d}\tau < u_0(x_2) + \int_0^t \left(I_0^+(x_2,\tau) - I_0^-(x_2,\tau) \right) \mathrm{d}\tau$$

from which follows $u_0(x_1) < u_0(x_2)$. On the other hand, if we assume $u_0(x_1) < u_0(x_2)$, then we have

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$$u_1(x_2,t) = u_0(x_2) + \int_0^t \left(I_0^+(x_2,\tau) - I_0^-(x_2,\tau)\right) d\tau = u_0(x_2) + \int_0^t \left(I_0^+(x_1,\tau) - I_0^-(x_1,\tau)\right) d\tau$$
$$+ \int_0^t \left[\int_{A^+u_0(x_1)} \lambda_1(\xi,\tau)(u_0(x_1) - u_0(x_2)) d\xi + \int_{A^-u_0(x_1)} \lambda_2(\xi,\tau)(u_0(x_1) - u_0(x_2)) d\xi\right] d\tau$$
$$- \int_0^t \left[\int_{A^+u_0(x_1) - A^+u_0(x_2)} \lambda_1(\xi,\tau)\,(u_0(\xi) - u_0(x_2))\, d\xi\right.$$
$$\left. - \int_{A^-u_0(x_2) - A^-u_0(x_1)} \lambda_2(\xi,\tau)\,(u_0(x_2) - u_0(\xi))\, d\xi\right] d\tau.$$

By observing that

$$\xi \in A^+u_0(x_1) - A^+u_0(x_2) \Leftrightarrow u_0(x_1) < u_0(\xi) \le u_0(x_2) \Rightarrow u_0(\xi) - u_0(x_2) \le 0;$$

and

$$\xi \in A^-u_0(x_2) - A^-u_0(x_1) \Leftrightarrow u_0(x_1) < u_0(\xi) \le u_0(x_2) \Rightarrow u_0(x_2) - u_0(\xi) \ge 0$$

from the previous equality it follows that

$$u_1(x_2,t) - u_1(x_1,t) \ge (u_0(x_2) - u_0(x_1)) \left[1 - \int_0^t \left(\int_{A^+u_0(x_1)} \lambda_1(\xi,\tau) d\xi + \int_{A^-u_0(x_1)} \lambda_2(\xi,\tau) d\xi\right) d\tau\right]$$
$$\ge (u_0(x_2) - u_0(x_1))(1 - Mt) > 0.$$

Hence $u_1(x_2,t) > u_1(x_1,t)$. An important consequence of (A) and (D) is that

$$A^+u_0(x) = A^+u_1(x,t); \qquad A^-u_0(x) = A^-u_1(x,t) \quad \forall x, \forall t \in \left[0, \frac{1}{M}\right[.$$

It is straightforward to extend by induction the properties that we have deduced for u_1 to all the functions of the sequence $\{u_n\}_{n=0}^{\infty}$. In particular, we have

$$0 < {u_0'}^2(x)(1 - Mt) \le u_0'(x) \frac{\partial u_n}{\partial x}(x,t) \le {u_0'}^2(x) \quad \forall x, \forall t \in \left[0, \frac{1}{M}\right[. \tag{3.5}$$

and

(A′) if $u_0(x_1) = u_0(x_2)$ then $u_n(x_1,t) = u_n(x_2,t)$, $\forall t \in \left[0, \frac{1}{M}\right]$;

(B′) $\forall x \in [0,1]$, $\forall t \in \left[0, \frac{1}{M}\right[$, $u_0'(x)$ and $\partial_x u_n(x,t)$ have the same sign;

(C′) if $u_0(x_1) < u_0(x_2)$ then $u_n(x_2,t) - u_n(x_1,t) \le u_0(x_2) - u_0(x_1)$, $\forall t \in \left[0, \frac{1}{M}\right[$;

(D′) $\forall x_1, x_2 \in [0,1]$, $\forall t \in \left[0, \frac{1}{M}\right[$, $u_0(x_1) < u_0(x_2) \Longleftrightarrow u_n(x_1,t) < u_n(x_2,t)$.

From (A′) and (D′) it follows

$$A^+u_0(x) = A^+u_n(x,t); \quad A^-u_0(x) = A^-u_n(x,t) \quad \forall x, \forall t \in \left[0, \frac{1}{M}\right[.$$

Finally, as in Theorem 2.1, we have

$$|u_{n+1}(x,t) - u_n(x,t)| \le 2\,\|u_n - u_{n-1}\|_\infty\, Mt$$

and

$$\left|\frac{\partial u_{n+1}}{\partial x}(x,t) - \frac{\partial u_n}{\partial x}(x,t)\right| \le Mt \left\|\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right\|_\infty$$

for all $n \in \mathbb{N}$, $x \in [0,1]$, $t \in [0, 1/2M]$. Hence there exists a function $u : [0,1] \times [0, \frac{1}{2M}] \to \mathbb{R}$ such that the sequence $\{u_n\}_{n=0}^{\infty}$ is uniformly convergent to u, $\{\partial u_n/\partial x\}_{n=0}^{\infty}$ is uniformly convergent to $\partial u/\partial x$, and $A^+u(x,t) = A^+u_0(x)$.

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The limit function, of course, satisfies the equation

$$u(x,t) = u_0(x) + \int_0^t \left[\int_{A^+u(x)} \lambda_1(\xi,\tau)(u(\xi,\tau) - u(x,\tau))\mathrm{d}\xi - \int_{A^-u(x)} \lambda_2(\xi,\tau)(u(x,\tau) - u(\xi,\tau))\mathrm{d}\xi \right] \mathrm{d}\tau.$$

As for Theorem 2.1 the proof may be iterated n times using $u(x, nT_0)$ as initial data, and this extends the validity of the solution to all positive times. □

Remark 3.2. If λ_1, λ_2 are continuous in time, then the function u of Theorem 3.1 is such that $\partial u/\partial t$ exists for all $(x,t) \in [0,1] \times [0, \frac{1}{2M}]$, and we have a result of existence for the Eq. (1.4) with initial condition $u(x,0) = u_0(x)$. We also notice that the expression (2.3) for $\partial u/\partial x$ is also valid under the hypothesis of Theorem 3.1.

Having established the existence of solutions of Eq. (1.4), the following lemma states one of their foremost properties, that of being non-growing, when regularity of solutions is assumed. The proof follows the technique commonly used for parabolic evolution equations [3].

Lemma 3.3 (*Maximum Principle*). *Let $u : [0,1] \times [0,T] \to \mathbb{R}$ be a regular solution of Eq.* (1.4)*, for some constant $T > 0$, with initial data $u(x,0) = u_0(x)$, where $u_0 : [0,1] \to \mathbb{R}$ is continuous. Let u_M and u_m be, respectively, the maximum and the minimum of the initial data u_0. Then, $\forall t \in [0,T_0]$, $\forall x \in [0,1]$*

$$u_m \le u(x,t) \le u_M.$$

Proof. Let us define a function u^ϵ such that $u^\epsilon(x,t) = u(x,t) - \epsilon t$, with $\epsilon > 0$. Let us assume that there exists x_0, t_0 such that $u^\epsilon(x_0,t_0) = \max(u^\epsilon)$ with $t_0 > 0$. Then we have

$$\frac{\partial u^\epsilon}{\partial t}(x_0,t_0) = \int_{A^+u(x_0,t_0)} \lambda_1(\xi,t)\,(u(\xi,t_0) - u(x,t_0))\,\mathrm{d}\xi - \int_{A^-u(x_0,t_0)} \lambda_2(\xi,t_0)\,(u(x_0,t_0) - u(\xi,t_0))\,\mathrm{d}\xi - \epsilon.$$

Q2

Our assumption implies $u(x_0,t_0) = \max(u(\cdot,t_0))$ from which it follows $A^+u(x_0,t_0) = \emptyset$, $A^-u(x_0,t_0) = [0,1]$. Then, after adding and subtracting ϵt_0 in the surviving integral, we have

$$\frac{\partial u^\epsilon}{\partial t}(x_0,t_0) + \int_0^1 \lambda_2(\xi,t_0)\left(u^\epsilon(x_0,t_0) - u^\epsilon(\xi,t_0)\right)\mathrm{d}\xi = -\epsilon,$$

which is a contradiction, because the left-hand side is, by construction, non-negative, and the right-hand side is strictly negative. Thus u^ϵ attains its maximum at $t = 0$, where it has to be $\max(u^\epsilon) = u_M$. Taking the limit $\epsilon \to 0$ it follows that $u(x,t) \le u_M$. Finally, defining $u^\epsilon(x,t) = u(x,t) + \epsilon t$ and following the same steps as above, one obtains $u_m \le u(x,t)$. □

In a similar way it is possible to prove the following theorem, which guarantees the unicity of the solution.

Theorem 3.4. *Let $u : [0,1] \times [0,T] \to \mathbb{R}$ be a regular solution of Eq.* (1.4)*, for some constant $T > 0$, with initial data $u(x,0) = u_0(x)$, where $u_0 : [0,1] \to \mathbb{R}$ is continuous. Then u is the only solution of Eq.* (1.4) *with initial data u_0.*

Proof. Let us assume that $v : [0,1] \times [0,T] \to \mathbb{R}$ is a solution of Eq. (1.4). Then we define a function $\psi : [0,1] \times [0,T] \to \mathbb{R}$ as $\psi(x,t) = v(x,t) - u(x,t)$, and we have

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= \int_{A^+v(x,t)\cap A^+u(x,t)} \lambda_1(\xi,t)\,(\psi(\xi,t) - \psi(x,t))\,\mathrm{d}\xi + \int_{A^+v(x,t)-A^+u(x,t)} \lambda_1(\xi,t)\,(v(\xi,t) - v(x,t))\,\mathrm{d}\xi \\ &\quad - \int_{A^+u(x,t)-A^+v(x,t)} \lambda_1(\xi,t)\,(u(\xi,t) - u(x,t))\,\mathrm{d}\xi - \int_{A^-v(x,t)\cap A^-u(x,t)} \lambda_2(\xi,t)\,(\psi(x,t) - \psi(\xi,t))\,\mathrm{d}\xi \\ &\quad - \int_{A^-v(x,t)-A^-u(x,t)} \lambda_2(\xi,t)\,(v(x,t) - v(\xi,t))\,\mathrm{d}\xi + \int_{A^-u(x,t)-A^-v(x,t)} \lambda_2(\xi,t)\,(u(x,t) - u(\xi,t))\,\mathrm{d}\xi.\end{aligned}$$

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We observe that the following relationships hold

$$\xi \in A^+v(x,t) - A^+u(x,t) \Leftrightarrow v(\xi,t) > v(x,t), \quad u(\xi,t) \le u(x,t);$$
$$\xi \in A^+u(x,t) - A^+v(x,t) \Leftrightarrow v(\xi,t) \le v(x,t), \quad u(\xi,t) > u(x,t);$$
$$\xi \in A^-v(x,t) - A^-u(x,t) \Leftrightarrow v(\xi,t) \le v(x,t), \quad u(\xi,t) > u(x,t);$$
$$\xi \in A^-u(x,t) - A^-v(x,t) \Leftrightarrow v(\xi,t) > v(x,t), \quad u(\xi,t) \le u(x,t).$$

Then we have

$$\frac{\partial \psi}{\partial t} \le \int_{A^+v(x,t)} \lambda_1(\xi,t)\,(\psi(\xi,t) - \psi(x,t))\,\mathrm{d}\xi - \int_{A^-u(x,t)} \lambda_2(\xi,t)\,(\psi(x,t) - \psi(\xi,t))\,\mathrm{d}\xi \tag{3.6}$$

and

$$\frac{\partial \psi}{\partial t} \ge \int_{A^+u(x,t)} \lambda_1(\xi,t)\,(\psi(\xi,t) - \psi(x,t))\,\mathrm{d}\xi - \int_{A^-v(x,t)} \lambda_2(\xi,t)\,(\psi(x,t) - \psi(\xi,t))\,\mathrm{d}\xi. \tag{3.7}$$

In particular, if (x_M, t_M) is a point where ψ attains its maximum, then $\partial_t \psi(x_M, t_M) \le 0$, and if (x_m, t_m) is a point where ψ attains its minimum, then $\partial_t \psi(x_m, t_m) \ge 0$. Then, with the help of the functions $\psi^M(x,t) = \psi(x,t) - \epsilon t$ and $\psi^m(x,t) = \psi(x,t) + \epsilon t$, it is straightforward to show, as in Lemma 3.3 that the inequality (3.6) implies $\psi(x,t) \le \max(\psi(x,0))$, and that the inequality (3.7) implies $\psi(x,t) \ge \min(\psi(x,0))$. But $\psi(x,0) = 0$, from which follows $u(x,t) = v(x,t)$ for any $(x,t) \in [0,1] \times [0,T]$. □

4. An explicit expression of the solution

The proof of Theorem 3.1 shows that, when the initial data belongs to $CF([0,1])$, it is $A^\pm u_0(x) = A^\pm u_n(x,t) = A^\pm u(x,t)$ for all $t \in [0,T]$. Or, in other words, we have that the cut functions α_j are completely determined by the choice of the initial data u_0, and do not depend on time. This fact leads to an explicit solution of Eq. (1.4), at least for some particular cases. In the following lemma we show that if one uses $A^\pm u_0(x)$ in place of $A^\pm u(x,t)$ in the definition (1.4), then the resulting modified equation, together with regular initial data, is easily solved.

Lemma 4.1. *Assume $u_0 \in C^2([0,1])$ and let λ_1, λ_2 be two real continuous functions defined on $[0,1] \times \mathbb{R}^+$. The solution of the equation*

$$\frac{\partial u}{\partial t}(x,t) = \int_{A^+u_0(x)} \lambda_1(\xi,t)\,(u(\xi,t) - u(x,t))\,\mathrm{d}\xi - \int_{A^-u_0(x)} \lambda_2(\xi,t)\,(u(x,t) - u(\xi,t))\,\mathrm{d}\xi \tag{4.1}$$

subject to the condition $u(x,0) = u_0(x)$ is

$$u(x,t) = u(0,t) + \int_0^x u_0'(\xi) \exp\left(-\int_0^t Y(\xi,\tau)\mathrm{d}\tau\right) \mathrm{d}\xi. \tag{4.2}$$

and

$$u(0,t) = u_0(0) + \int_0^t \Bigg[\int_{A^+u_0(0)} \lambda_1(\xi,\tau) \int_0^\xi u_0'(\zeta) \exp\left(-\int_0^\tau Y(\zeta,\sigma)\mathrm{d}\sigma\right) \mathrm{d}\zeta\,\mathrm{d}\xi$$
$$- \int_{A^-u_0(0)} \lambda_2(\xi,\tau) \int_0^\xi u_0'(\zeta) \exp\left(-\int_0^\tau Y(\zeta,\sigma)\mathrm{d}\sigma\right) \mathrm{d}\zeta\,\mathrm{d}\xi\Bigg]\mathrm{d}\tau \tag{4.3}$$

where we defined

$$Y(x,t) = \int_{A^+u_0(x)} \lambda_1(\xi,t)\mathrm{d}\xi + \int_{A^-u_0(x)} \lambda_2(\xi,t)\mathrm{d}\xi.$$

Proof. By deriving Eq. (4.1) partially with respect to x, we have

$$\frac{\partial^2 u}{\partial t \partial x}(x,t) = -\frac{\partial u}{\partial x}(x,t) Y(x,t). \tag{4.4}$$

We observe, that Y is a function of x and t only, and does not depend on u. Then it is straightforward to integrate (4.4) with respect to time, to obtain

$$\frac{\partial u}{\partial x}(x,t) = u_o'(x) \exp\left(-\int_0^t Y(x,\tau)\mathrm{d}\tau\right)$$

from which follows (4.2). We determine the value of $u(0,t)$ by inserting the expression (4.2) in the Eq. (4.1). The result may immediately be integrated, yielding (4.3). □

Remark 4.2. Expression (4.2) together with (4.3) shows that the solution of Eq. (4.1) exists for all times where $\int_0^t Y(x,\tau)\mathrm{d}\tau$ exists. Furthermore, if λ_1, λ_2 are chosen such that $Y(x,t) \geq \sigma > 0$, for some positive constant σ and for all $(x,t) \in [0,1] \times [0,\infty)$, then (4.2) implies $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} u(0,t)$.

Remark 4.3. If, for some choice of λ_1, λ_2 and u_0 the function u defined by expressions (4.2) and (4.3) has the property that $A^{\pm}u(x,t) = A^{\pm}u(x,0)$, then u is also the solution of Eq. (1.4), for the same choice of λ_1, λ_2, and with initial data $u(x,0) = u_0(x)$.

Remark 4.4. From Theorem 2.1 we know that the solution of Eq. (1.4) with monotonic initial data has the property that $A^{\pm}u(x,t) = A^{\pm}u(x,0)$. Then expressions (4.2) and (4.3) give the explicit solution for this special case.

5. Some open problems

A: It would be interesting to find the class of all triplets $(\lambda_1, \lambda_2, u_0)$ that satisfy the condition of Remark 4.3. For this class expressions (4.2) and (4.3) give the explicit solution of Eq. (1.4).

B: An obvious problem is that of establishing local and global existence results for more general classes of initial data, such as C^0 functions. It seems interesting to investigate the case of a sequence of initial data having a growing and unbounded number of oscillations in [0, 1 :].

C: It is also of interest to establish local and global existence and uniqueness theorems for the following more general, one-dimensional equation:

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) = {} & \frac{1}{\left|A^+u(x,t)\right|}\int_{A^+u(x,t)} \lambda_1(x,\xi,t)\mathrm{e}^{-c_1\tau(x-\xi)^2}(u(\xi,t)-u(x,t))\mathrm{d}\xi \\ & - \frac{1}{\left|A^-u(x,t)\right|}\int_{A^-u(x,t)} \lambda_2(x,\xi,t)\mathrm{e}^{-c_2\tau(x-\xi)^2}(u(x,t)-u(\xi,t))\mathrm{d}\xi,\end{aligned}$$

where $c_i > 0$.

D: Eq. (1.4) may be extended to more than one spatial dimension as follows. Let $\Omega \subset \mathbb{R}^N$ be an open regular bounded set, $\lambda_1, \lambda_2 : \Omega \times \mathbb{R}^+ \to [0,\infty)$ regular bounded functions, and $u_0 : \bar{\Omega} \to \mathbb{R}$. Given a function $u : \bar{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ we define $A^+u(x,t) = \{\xi \in \bar{\Omega} \mid u(\xi,t) > u(x,t)\}$ and $A^-u(x,t) = \bar{\Omega} - A^+u(x,t)$. Then we ask for existence and unicity results for Eq. (1.4), with the newly defined sets $A^{\pm}u(x,t)$, subject to the initial data $u(x,0) = u_0$.

To our knowledge no results are known for these problems.

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